

# HERMITE-HADAMARD-FEJER TYPE INEQUALITIES FOR CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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**ABSTRACT.** In this paper, firstly we have established Hermite–Hadamard–Fejér inequality for fractional integrals. Secondly, an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for the fractional integrals have been obtained. The some results presented here would provide extensions of those given in earlier works.

## 1. INTRODUCTION

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . The inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

is well known in the literature as Hermite–Hadamard’s inequality [4].

The most well-known inequalities related to the integral mean of a convex function  $f$  are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities.

In [3], Fejér established the following Fejér inequality which is the weighted generalization of Hermite–Hadamard inequality (1.1):

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be convex function. Then the inequality*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx$$

*holds, where  $g : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable and symmetric to  $(a+b)/2$ .*

For some results which generalize, improve, and extend the inequalities (1.1) and (1.2) see [1, 5, 6, 7, 11, 15].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

**Definition 1.** *Let  $f \in L[a, b]$ . The Riemann–Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

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$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively, where  $\Gamma(\alpha)$  is the Gamma function defined by  $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$

and  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [2, 8, 9, 13, 14, 16, 17].

In [13], Sarıkaya et. al. represented Hermite-Hadamard's inequalities in fractional integral forms as follows.

**Theorem 2.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq a < b$  and  $f \in L[a, b]$ . If  $f$  is a convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold

$$(1.3) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{f(a) + f(b)}{2}$$

with  $\alpha > 0$ .

In [13] some Hermite-Hadamard type integral inequalities for fractional integral were proved using the following lemma.

**Lemma 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f' \in L[a, b]$  then the following equality for fractional integrals holds:

$$(1.4) \quad \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] = \frac{b-a}{2} \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta + (1-t)b) dt.$$

**Theorem 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$  then the following inequality for fractional integrals holds:

$$(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right| \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) [|f'(a)| + |f'(b)|].$$

**Lemma 2** ([10, 17]). For  $0 < \alpha \leq 1$  and  $0 \leq a < b$ , we have

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

In this paper, we firstly represented Hermite-Hadamard-Fejér inequality in fractional integral forms which is the weighted generalization of Hermite-Hadamard inequality (1.3). Secondly, we obtained some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for the fractional integrals.

## 2. MAIN RESULTS

Throughout this section, let  $\|g\|_{\infty} = \sup_{t \in [a, b]} |g(x)|$ , for the continuous function  $g : [a, b] \rightarrow \mathbb{R}$ .

**Lemma 3.** *If  $g : [a, b] \rightarrow \mathbb{R}$  is integrable and symmetric to  $(a + b)/2$  with  $a < b$ , then*

$$J_{a+}^{\alpha} g(b) = J_{b-}^{\alpha} g(a) = \frac{1}{2} [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)]$$

with  $\alpha > 0$ .

*Proof.* Since  $g$  is symmetric to  $(a + b)/2$ , we have  $g(a + b - x) = g(x)$ , for all  $x \in [a, b]$ . Hence, in the following integral setting  $x = tb + (1 - t)a$  and  $dx = (b - a) dt$  gives

$$\begin{aligned} J_{a+}^{\alpha} g(b) &= \frac{1}{\Gamma(\alpha)} \int_a^b (b - x)^{\alpha-1} g(x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (x - a)^{\alpha-1} g(a + b - x) dx \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b (x - a)^{\alpha-1} g(x) dx = J_{b-}^{\alpha} g(a). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 4.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be convex function with  $a < b$  and  $f \in L[a, b]$ . If  $g : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable and symmetric to  $(a + b)/2$ , then the following inequalities for fractional integrals hold*

$$\begin{aligned} (2.1) \quad f\left(\frac{a+b}{2}\right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] &\leq [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \\ &\leq \frac{f(a) + f(b)}{2} [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] \end{aligned}$$

with  $\alpha > 0$ .

*Proof.* Since  $f$  is a convex function on  $[a, b]$ , we have for all  $t \in [0, 1]$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta + (1-t)b + tb + (1-t)a}{2}\right) \\ (2.2) \quad &\leq \frac{f(ta + (1-t)b) + f(tb + (1-t)a)}{2}. \end{aligned}$$

Multiplying both sides of (2.2) by  $2t^{\alpha-1}g(tb + (1-t)a)$  then integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right) \int_0^1 t^{\alpha-1} g(tb + (1-t)a) dt \\ &\leq \int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f(tb + (1-t)a)] g(tb + (1-t)a) dt \\ &= \int_0^1 t^{\alpha-1} f(ta + (1-t)b) g(tb + (1-t)a) dt + \int_0^1 t^{\alpha-1} f(tb + (1-t)a) g(tb + (1-t)a) dt. \end{aligned}$$

Setting  $x = tb + (1 - t)a$ , and  $dx = (b - a) dt$  gives

$$\begin{aligned}
& \frac{2}{(b-a)^\alpha} f\left(\frac{a+b}{2}\right) \int_a^b (x-a)^{\alpha-1} g(x) dx \\
& \leq \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (x-a)^{\alpha-1} f(a+b-x) g(x) dx + \int_0^1 (x-a)^{\alpha-1} f(x) g(x) dx \right\} \\
& = \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (b-x)^{\alpha-1} f(x) g(a+b-x) dx + \int_0^1 (x-a)^{\alpha-1} f(x) g(x) dx \right\} \\
& = \frac{1}{(b-a)^\alpha} \left\{ \int_a^b (b-x)^{\alpha-1} f(x) g(x) dx + \int_0^1 (x-a)^{\alpha-1} f(x) g(x) dx \right\}.
\end{aligned}$$

Therefore, by Lemma 3 we have

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} f\left(\frac{a+b}{2}\right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)]$$

and the first inequality is proved.

For the proof of the second inequality in (2.1) we first note that if  $f$  is a convex function, then, for all  $t \in [0, 1]$ , it yields

$$(2.3) \quad f(ta + (1-t)b) + f(tb + (1-t)a) \leq f(a) + f(b).$$

Then multiplying both sides of (2.3) by  $2t^{\alpha-1}g(tb + (1-t)a)$  and integrating the resulting inequality with respect to  $t$  over  $[0, 1]$ , we obtain

$$\begin{aligned}
& \int_0^1 t^{\alpha-1} f(ta + (1-t)b) g(tb + (1-t)a) dt + \int_0^1 t^{\alpha-1} f(tb + (1-t)a) g(tb + (1-t)a) dt \\
& \leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} g(tb + (1-t)a) dt
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(b-a)^\alpha} [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha} \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)]$$

The proof is completed.  $\square$

**Remark 1.** In Theorem 4,

- (i) if we take  $\alpha = 1$ , then inequality (2.1) becomes inequality (1.2) of Theorem 1.
- (ii) if we take  $g(x) = 1$ , then inequality (2.1) becomes inequality (1.3) of Theorem 2.

**Lemma 4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$  and  $f' \in L[a, b]$ . If  $g : [a, b] \rightarrow \mathbb{R}$  is integrable and symmetric to  $(a+b)/2$  then the following equality for fractional integrals holds

$$\begin{aligned}
(2.4) \quad & \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \\
& = \frac{1}{\Gamma(\alpha)} \int_a^b \left[ \int_a^t (b-s)^{\alpha-1} g(s) ds - \int_t^b (s-a)^{\alpha-1} g(s) ds \right] f'(t) dt
\end{aligned}$$

with  $\alpha > 0$ .

*Proof.* It suffices to note that

$$\begin{aligned}
I &= \int_a^b \left[ \int_a^t (b-s)^{\alpha-1} g(s) ds - \int_t^b (s-a)^{\alpha-1} g(s) ds \right] f'(t) dt \\
&= \int_a^b \left( \int_a^t (b-s)^{\alpha-1} g(s) ds \right) f'(t) dt + \int_a^b \left( - \int_t^b (s-a)^{\alpha-1} g(s) ds \right) f'(t) dt \\
&= I_1 + I_2.
\end{aligned}$$

By integration by parts and Lemma 3 we get

$$\begin{aligned}
I_1 &= \left( \int_a^t (b-s)^{\alpha-1} g(s) ds \right) f(t) \Big|_a^b - \int_a^b (b-t)^{\alpha-1} g(t) f(t) dt \\
&= \left( \int_a^b (b-s)^{\alpha-1} g(s) ds \right) f(b) - \int_a^b (b-t)^{\alpha-1} (fg)(t) dt \\
&= \Gamma(\alpha) [f(b) J_{a+}^\alpha g(b) - J_{a+}^\alpha (fg)(b)] \\
&= \Gamma(\alpha) \left[ \frac{f(b)}{2} [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - J_{a+}^\alpha (fg)(b) \right]
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \left( - \int_t^b (s-a)^{\alpha-1} g(s) ds \right) f(t) \Big|_a^b - \int_a^b (t-a)^{\alpha-1} g(t) f(t) dt \\
&= \left( \int_a^b (s-a)^{\alpha-1} g(s) ds \right) f(a) - \int_a^b (t-a)^{\alpha-1} (fg)(t) dt \\
&= \Gamma(\alpha) \left[ \frac{f(a)}{2} [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - J_{b-}^\alpha (fg)(a) \right].
\end{aligned}$$

Thus, we can write

$$I = I_1 + I_2 = \Gamma(\alpha) \left\{ \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right\}.$$

Multiplying the both sides by  $(\Gamma(\alpha))^{-1}$  we obtain (2.4) which completes the proof.  $\square$

**Remark 2.** In Lemma 4, if we take  $g(x) = 1$ , then equality (2.4) becomes equality (1.4) of Lemma 1.

**Theorem 5.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $f' \in L[a, b]$  with  $a < b$ . If  $|f'|$  is convex on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and symmetric to  $(a+b)/2$ , then the following inequality for fractional integrals holds

$$\begin{aligned}
&\left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\
(2.5) \leq &\frac{(b-a)^{\alpha+1} \|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)} \left( 1 - \frac{1}{2^\alpha} \right) [|f'(a)| + |f'(b)|]
\end{aligned}$$

with  $\alpha > 0$ .

*Proof.* From Lemma 4 we have

$$(2.6) \quad \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] - [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \right| \\ \leq \frac{1}{\Gamma(\alpha)} \int_a^b \left| \int_a^t (b-s)^{\alpha-1} g(s) ds - \int_t^b (s-a)^{\alpha-1} g(s) ds \right| |f'(t)| dt.$$

Since  $|f'|$  is convex on  $[a, b]$ , we know that for  $t \in [a, b]$

$$(2.7) \quad |f'(t)| = \left| f' \left( \frac{b-t}{b-a} a + \frac{t-a}{b-a} b \right) \right| \leq \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)|,$$

and since  $g : [a, b] \rightarrow \mathbb{R}$  is symmetric to  $(a+b)/2$  we write

$$\int_t^b (s-a)^{\alpha-1} g(s) ds = \int_a^{a+b-t} (b-s)^{\alpha-1} g(a+b-s) ds = \int_a^{a+b-t} (b-s)^{\alpha-1} g(s) ds,$$

then we have

$$(2.8) \quad \left| \int_a^t (b-s)^{\alpha-1} g(s) ds - \int_t^b (s-a)^{\alpha-1} g(s) ds \right| \\ = \left| \int_t^{a+b-t} (b-s)^{\alpha-1} g(s) ds \right| \\ \leq \begin{cases} \int_t^{a+b-t} |(b-s)^{\alpha-1} g(s)| ds & t \in [a, \frac{a+b}{2}] \\ \int_{a+b-t}^t |(b-s)^{\alpha-1} g(s)| ds & t \in [\frac{a+b}{2}, b] \end{cases}.$$

A combination of (2.6), (2.7) and (2.8), we get

$$\left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] - [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \right| \\ \leq \frac{1}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left( \int_t^{a+b-t} |(b-s)^{\alpha-1} g(s)| ds \right) \left( \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\ + \frac{1}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left( \int_{a+b-t}^t |(b-s)^{\alpha-1} g(s)| ds \right) \left( \frac{b-t}{b-a} |f'(a)| + \frac{t-a}{b-a} |f'(b)| \right) dt \\ \leq \frac{\|g\|_{\infty}}{(b-a)\Gamma(\alpha+1)} \left\{ \int_a^{\frac{a+b}{2}} [(b-t)^{\alpha} - (t-a)^{\alpha}] ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \right. \\ (2.9) \quad \left. \int_{\frac{a+b}{2}}^b [(t-a)^{\alpha} - (b-t)^{\alpha}] ((b-t)|f'(a)| + (t-a)|f'(b)|) dt \right\}$$

Since

$$(2.10) \quad \int_a^{\frac{a+b}{2}} [(b-t)^{\alpha} - (t-a)^{\alpha}] (b-t) dt \\ = \int_{\frac{a+b}{2}}^b [(t-a)^{\alpha} - (b-t)^{\alpha}] (t-a) dt \\ = \frac{(b-a)^{\alpha+2}}{(\alpha+1)} \left( \frac{\alpha+1}{\alpha+2} - \frac{1}{2^{\alpha+1}} \right)$$

and

$$\begin{aligned}
 & \int_a^{\frac{a+b}{2}} [(b-t)^\alpha - (t-a)^\alpha] (t-a) dt \\
 &= \int_{\frac{a+b}{2}}^b [(t-a)^\alpha - (b-t)^\alpha] (b-t) dt \\
 (2.11) \quad &= \frac{(b-a)^{\alpha+2}}{(\alpha+1)} \left( \frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}} \right)
 \end{aligned}$$

Hence, if we use (2.10) and (2.11) in (2.9), we obtain the desired result. This completes the proof.  $\square$

**Remark 3.** In Theorem 5, if we take  $g(x) = 1$ , then equality (2.5) becomes equality (1.5) of Theorem 3.

**Theorem 6.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $f' \in L[a, b]$  with  $a < b$ . If  $|f'|^q, q > 1$ , is convex on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and symmetric to  $(a+b)/2$ , then the following inequality for fractional integrals holds

$$\begin{aligned}
 (2.12) \quad & \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\
 & \leq \frac{2(b-a)^{\alpha+1} \|g\|_\infty}{(b-a)^{1/q} (\alpha+1) \Gamma(\alpha+1)} \left( 1 - \frac{1}{2^\alpha} \right) \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}
 \end{aligned}$$

where  $\alpha > 0$  and  $1/p + 1/q = 1$ .

*Proof.* Using Lemma 4, Hölder's inequality, (2.8) and the convexity of  $|f'|^q$ , it follows that

$$\begin{aligned}
 & \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\
 & \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left| \int_t^{a+b-t} (b-s)^{\alpha-1} g(s) ds \right| dt \right)^{1-1/q} \left( \int_a^b \left| \int_t^{a+b-t} (b-s)^{\alpha-1} g(s) ds \right| |f'(t)|^q dt \right)^{1/q} \\
 & \leq \frac{1}{\Gamma(\alpha)} \left[ \int_a^{\frac{a+b}{2}} \left( \int_t^{a+b-t} |(b-s)^{\alpha-1} g(s)| ds \right) dt + \int_{\frac{a+b}{2}}^b \left( \int_{a+b-t}^t |(b-s)^{\alpha-1} g(s)| ds \right) dt \right]^{1-1/q} \\
 & \quad \times \left[ \int_a^{\frac{a+b}{2}} \left( \int_t^{a+b-t} |(b-s)^{\alpha-1} g(s)| ds \right) |f'(t)|^q dt + \int_{\frac{a+b}{2}}^b \left( \int_{a+b-t}^t |(b-s)^{\alpha-1} g(s)| ds \right) |f'(t)|^q dt \right]^{1/q} \\
 & \leq \frac{2^{1-1/q} \|g\|_\infty}{(b-a)^{1/q} \Gamma(\alpha+1)} \left( \frac{(b-a)^{\alpha+1}}{\alpha+1} \left[ 1 - \frac{1}{2^\alpha} \right] \right)^{1-1/q} \\
 & \quad \times \left\{ \int_a^{\frac{a+b}{2}} [(b-t)^\alpha - (t-a)^\alpha] ((b-t)|f'(a)|^q + (t-a)|f'(b)|^q) dt \right. \\
 (2.13) \quad & \left. + \int_{\frac{a+b}{2}}^b [(t-a)^\alpha - (b-t)^\alpha] ((b-t)|f'(a)|^q + (t-a)|f'(b)|^q) dt \right\}^{1/q}
 \end{aligned}$$

where it is easily seen that

$$\begin{aligned} & \int_a^{\frac{a+b}{2}} \left( \int_t^{a+b-t} (b-s)^{\alpha-1} ds \right) dt + \int_{\frac{a+b}{2}}^b \left( \int_{a+b-t}^t (b-s)^{\alpha-1} ds \right) dt \\ &= \frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)} \left[ 1 - \frac{1}{2^\alpha} \right]. \end{aligned}$$

Hence, if we use (2.10) and (2.11) in (2.13), we obtain the desired result. This completes the proof.  $\square$

We can state another inequality for  $q > 1$  as follows:

**Theorem 7.** *Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $f' \in L[a, b]$  with  $a < b$ . If  $|f'|^q, q > 1$ , is convex on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  is continuous and symmetric to  $(a+b)/2$ , then the following inequalities for fractional integrals hold:*

(i)

$$\begin{aligned} & \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\ (2.14) \leq & \frac{2^{1/p} \|g\|_\infty (b-a)^{\alpha+1}}{(\alpha p + 1)^{1/p} \Gamma(\alpha + 1)} \left( 1 - \frac{1}{2^{\alpha p}} \right)^{1/p} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \end{aligned}$$

with  $\alpha > 0$ .

(ii)

$$\begin{aligned} & \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\ (2.15) \leq & \frac{\|g\|_\infty (b-a)^{\alpha+1}}{(\alpha p + 1)^{1/p} \Gamma(\alpha + 1)} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \end{aligned}$$

for  $0 < \alpha \leq 1$ . Where  $1/p + 1/q = 1$ .

*Proof.* (i) Using Lemma 4, Hölder's inequality, (2.8) and the convexity of  $|f'|^q$ , it follows that

$$\begin{aligned} & \left| \left( \frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\ & \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^b \left| \int_t^{a+b-t} (b-s)^{\alpha-1} g(s) ds \right|^p dt \right)^{1/p} \left( \int_a^b |f'(t)|^q dt \right)^{1/q} \\ & \leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)} \left( \int_a^{\frac{a+b}{2}} [(b-t)^\alpha - (t-a)^\alpha]^p dt + \int_{\frac{a+b}{2}}^b [(t-a)^\alpha - (b-t)^\alpha]^p dt \right)^{1/p} \\ & \quad \times \left( \int_a^b \left( \frac{b-t}{b-a} |f'(a)|^q + \frac{t-a}{b-a} |f'(b)|^q \right) dt \right)^{1/q} \\ & = \frac{\|g\|_\infty (b-a)^{\alpha+1}}{\Gamma(\alpha+1)} \left( \int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha]^p dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]^p dt \right)^{1/p} \\ (2.16) \quad & \times \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \end{aligned}$$



$$\begin{aligned}
&\leq \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} \left( \int_0^{\frac{1}{2}} [(1-t)^{\alpha p} - t^{\alpha p}] dt + \int_{\frac{1}{2}}^1 [t^{\alpha p} - (1-t)^{\alpha p}] dt \right)^{1/p} \\
&\quad \times \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q} \\
&\leq \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)} \left( \frac{2}{\alpha p + 1} \left[ 1 - \frac{1}{2^{\alpha p}} \right] \right)^{1/p} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}.
\end{aligned}$$

Here we use

$$[(1-t)^{\alpha} - t^{\alpha}]^p \leq (1-t)^{\alpha p} - t^{\alpha p}$$

for  $t \in [0, 1/2]$  and

$$[t^{\alpha} - (1-t)^{\alpha}]^p \leq t^{\alpha p} - (1-t)^{\alpha p}$$

for  $t \in [1/2, 1]$ , which follows from

$$(A - B)^q \leq A^q - B^q,$$

for any  $A \geq B \geq 0$  and  $q \geq 1$ . Hence the inequality (2.14) is proved.

(ii) The inequality (2.15) is easily proved using (2.16) and Lemma 2.  $\square$

**Remark 4.** In Theorem 7, if we take  $\alpha = 1$ , then equality (2.15) becomes equality in [17, Corollary 13].

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